ME 481 Fall 2023: Drs. Mehrdad Nejhad & Trevor Sorensen

FEM/FEA Boundary Conditions, and Failure Analysis

Three Cases must use FEM/FEA:

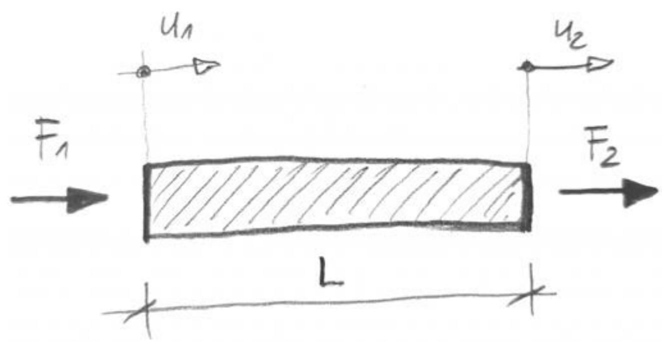
(1) Irregular Geometry
(2) Complex Material Properties
(3) Nonlinearity

NOTE: **Three general cases you would need to use FEA for your Analysis and Analytical Solutions do not apply**:

(1) Irregular Geometry,
(2) Complex and Non-homogenous Materials Properties, and
(3) Nonlinearity of the Problems

I: Classical 1-D FEA Problem (Axial Loading of a Bar)

https://enterfea.com/finite-element-analysis-by-hand/



$$\sigma = E \cdot \varepsilon$$

STRESS: $\sigma = \frac{F}{A}$ ← FORCE; A ← CROSS-SECTION AREA

YOUNG'S MODULUS

STRAIN: $\varepsilon = \frac{\Delta L}{L}$

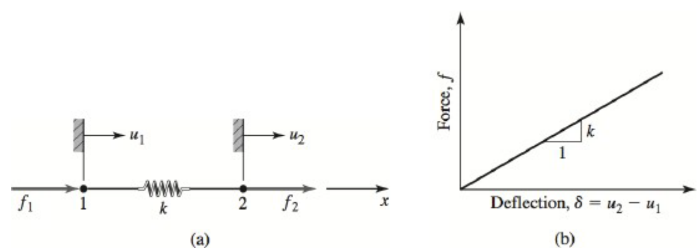
CHANGE IN LENGTH: $\Delta L = u_2 - u_1$

ORYGINAL LENGTH

$$\frac{F}{A} = \sigma = E \cdot \varepsilon = E \cdot \frac{\Delta L}{L} = E \cdot \frac{u_2 - u_1}{L}$$

$$F = \frac{A \cdot E}{L} (u_2 - u_1)$$

F = K ΔX , where **<u>K = AE/L</u>** & **ΔX = ΔU**



$$F_1 = -\frac{A \cdot E}{L} (u_2 - u_1) = \frac{A \cdot E}{L} (u_1 - u_2)$$

$$F_2 = \frac{A \cdot E}{L} (u_2 - u_1)$$

FORCES VECTOR STIFFNESS MATRIX DISPLACEMENTS VECTOR

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} \frac{A \cdot E}{L} & -\frac{A \cdot E}{L} \\ -\frac{A \cdot E}{L} & \frac{A \cdot E}{L} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

AE/L = K

$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}$$

$$\begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = [k_e] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \quad \text{with} \quad [k_e] = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$$

known $\{F\} = [K]\ \{X\}$ unknown

$$\{F\} = [K] \cdot \{U\}$$

Array of applied forces (one for each DOF)

Matrix of stiffnesses (DOF x DOF)

Array of displacements (one for each DOF)

Pre-Dot the above Eq by $[K]^{-1}$

$[K]^{-1}\ \{F\} = [K]^{-1}\ [K]\ \{U\}$, where $[K]^{-1}\ [K] = [1]$ ==➔ $[K]^{-1}\ \{F\} = \{U\}$

$$\{U_i\} = [K_{ij}]^{-1}\ \{F_j\}$$

$$\{\varepsilon_i\} = [d/dx_{ij}]\ \{u_j\}, \quad \{\sigma_i\}_{6x1} = [C_{ij}]_{6x6}\ \{\varepsilon_i\}_{6x1}$$

<u>General 3D Case:</u>

$\sigma_i = \sigma_x, \sigma_y, \sigma_z, \sigma_{xy}, \sigma_{yz}, \sigma_{zx}$ & $\sigma_{ij} = \sigma_{ji}$

$\varepsilon_i = \varepsilon_x, \varepsilon_y, \varepsilon_z, \varepsilon_{xy}, \varepsilon_{yz}, \varepsilon_{zx}$ & $\varepsilon_{ij} = \varepsilon_{ji}$

II: 3D General Materials Properties in Mechanices

C_{ij} = C_{ji} = **21** independent materials **Anisotropic properties** **[Cij]**

= **9** independent materials **Orthotropic properties (3E's, 3Gs, 3νs)**

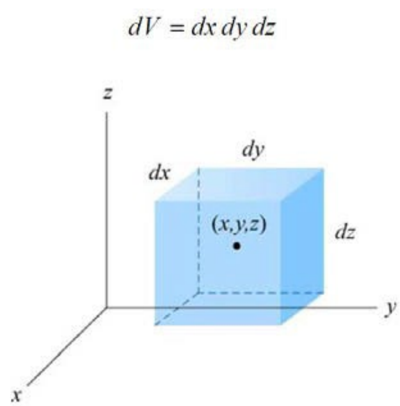
= **5** independent materials **Transversly-Isotropic properties (2Es, 1G, 2νs)**

= **2** independent materials **Isotropic properties (E & ν , since G = E/(2(1+ν)))**

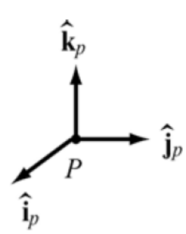
III. Geometries

For the following **Regular Geometries** Analytical Solutions and associated **Governing Equations** (written for the given coordinate system) plus the boundary and initial conditions can be used. Otherwise (i.e., Irregular Geometries) FEA should be used).

(III.1): Cartesian/Orthogonal Coordinate System: x, y, z



Volume element in Cartesian coordinates



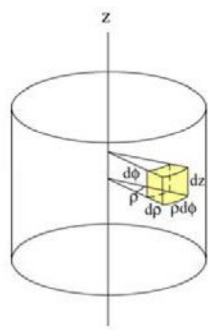
Unit Vectors

Heat Conduction Eq in Cartesian Coordinate System:

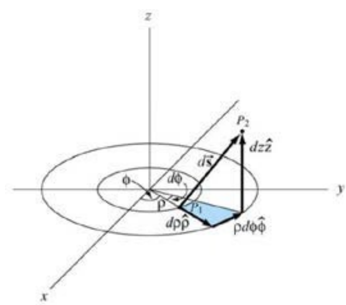
$$\frac{\partial^2 T}{\partial x^2} + \frac{\dot{e}_{\text{gen}}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

(III.2): Cylindrical/Orthogonal Coordinate System: ρ, ϕ, z

$$dV = \rho d\phi \, d\rho \, dz$$



Volume element for a cylinder



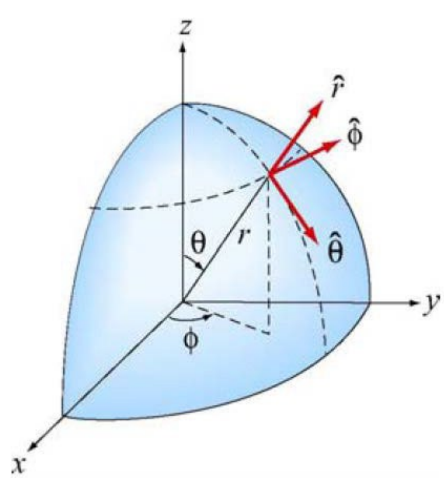
Unit Vectors

Heat Conduction Eq in Cylindrical Coordinate System:

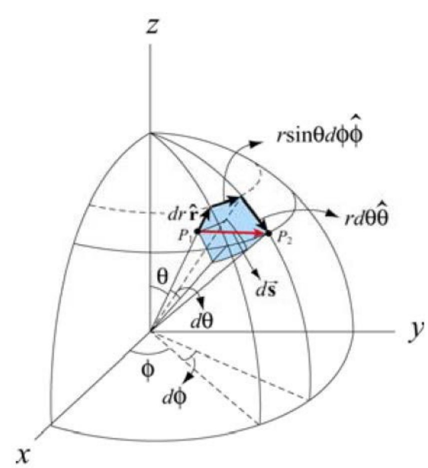
$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\dot{e}_{\text{gen}}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

(III.3): Spherical/Orthogonal Coordinate System: r, ϕ, θ

$$dV = r^2 \sin\theta \, d\theta \, d\phi \, dr$$



Spherical coordinates



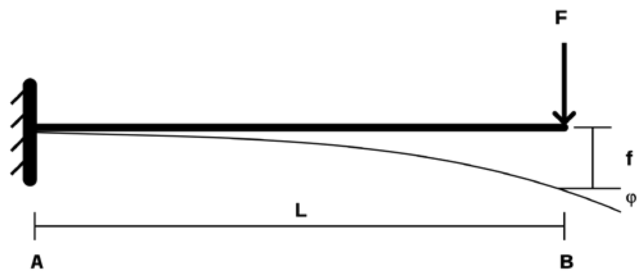
Unit Vectors

Heat Conduction Eq in Spherical Coordinate System:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{\dot{e}_{\text{gen}}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

IV. Nonlinearity:

IV.1. Nonlinearity due to Small Deformation versus Large Deformation



If **"*f*"** is the tip **Deflection** and **"*t*"** is the **thickness** of a beam, then:

If $f << t$, then f is considered "**small deformation**" and the problem is **Linear**.

If $f >> t$, then f is considered "**large deformation**" and the problem is **Nonlinear**.

IV.2. Nonlinearity due to the Governing Equations

If the **Field/Dependent Variable** within the **Individual Terms** of a **Governing Equation** is to the **first power** only, then the problem is **Linear**.

If the **Field/Dependent Variable** within the **Individual Terms** of a **Governing Equation** is to the **second or higher power**, then the problem is **Nonlinear**.

If K is variable in space, the 3D Heat Conduction Eq is:

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{e}_{\text{gen}}=\rho c\frac{\partial T}{\partial t}$$

If K is constant in space and the material is ***Isotropic***, the 3D Heat Conduction Eq is (where, $\alpha = k/\rho c$: Thermal Diffusivity)):

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{e}_{\text{gen}}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

If *K* is constant in space and the material is ***Orthotropic***, the 3D Heat Conduction Eq is:

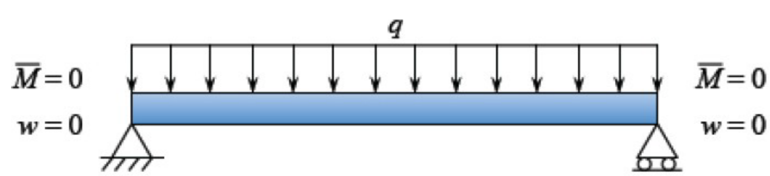
$$K_x\,(\partial^2 T/\partial x^2) + K_y\,(\partial^2 T/\partial y^2) + K_z\,(\partial^2 T/\partial z^2) + (e_{\text{gen}}) = \rho c\,(\partial T/\partial t)$$

The above Eqs are **Linear** since for each term of the Eq, the Field/Dependent Variable (i.e., ***T***) is to the **first power**.

If the Thermal Conductivity components are functions of Temperature, then the above equation becomes **Nonlinear** since it will have the Field/Dependent Variable (i.e., ***T***) to the **second power** for each term of the Eq. as:

$$K_x(T)\,(\partial^2 T/\partial x^2) + K_y(T)\,(\partial^2 T/\partial y^2) + K_z(T)\,(\partial^2 T/\partial z^2) + (e_{\text{gen}}) = \rho c\,(\partial T/\partial t)$$

Governing Eq for Bending of a Beam:

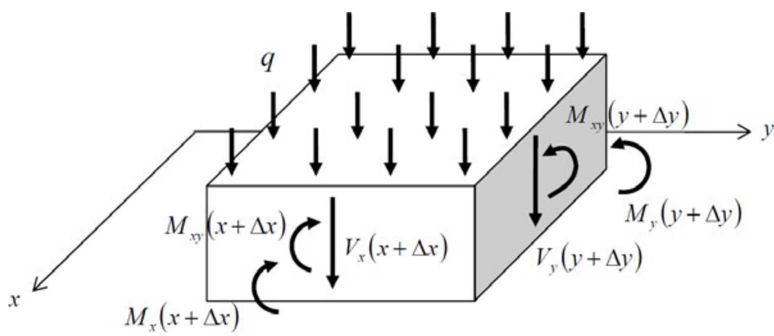


$$EI\frac{d^4 w}{dx^4} = q(x)$$

$$\frac{\mathrm{d}^3 w}{\mathrm{d}x^3} = \frac{qx}{EA} + C_1$$
$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = \frac{qx^2}{EA2} + C_1 x + C_2$$
$$\frac{\mathrm{d}w}{\mathrm{d}x} = \frac{qx^3}{EA6} + \frac{C_1 x^2}{2} + C_2 x + C_3$$
$$\frac{\mathrm{d}w}{\mathrm{d}x} = \frac{qx^4}{EA24} + \frac{C_1 x^3}{6} + \frac{C_2 x^2}{2} + C_3 x + C_4$$

$$w(x) = \frac{qx}{24EA}(l^3 - 2lx^2 + x^3)$$

Governing Eq for Bending of a Plate:



$$\boxed{\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = -\frac{q}{D}}$$

Where:

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

V. Finite Element Modeling & Analysis:

Any **Finite Element Modeling (FEM)** and **Finite Element Analysis (FEA)** includes the following three main steps: **(1) Pre-Processing**, **(2) Model Solution**, **(3) Post-Processing**.

V.1. Step 1: Pre-Processing

1. : V.1.1. Object Generation:

1. 3D: **3D** Solid Modeling with **Solid Elements**
2. 1D & 2D: **1D** **Beam Element** & **2D** **Shell Elements**

2. : V.1.2. Pick Analysis Type (for Step 2: Model Solution):

1. **Structural** Analysis (**Static** or **Dynamic**; **Linear** or **Non-Linear**)
2. **Heat Transfer** Analysis (**Steady State** or **Transient**; **Linear** or **Non-Linear**)
3. **Fluids** Analysis (**Steady State** or **Transient**; **Linear** or **Non-Linear**)

3. : V.1.3. Meshing (see Meshing details):

1. **Select Elements**
2. **Select Nodes**

4. : V.1.4. Material Properties (see [Cij] details):

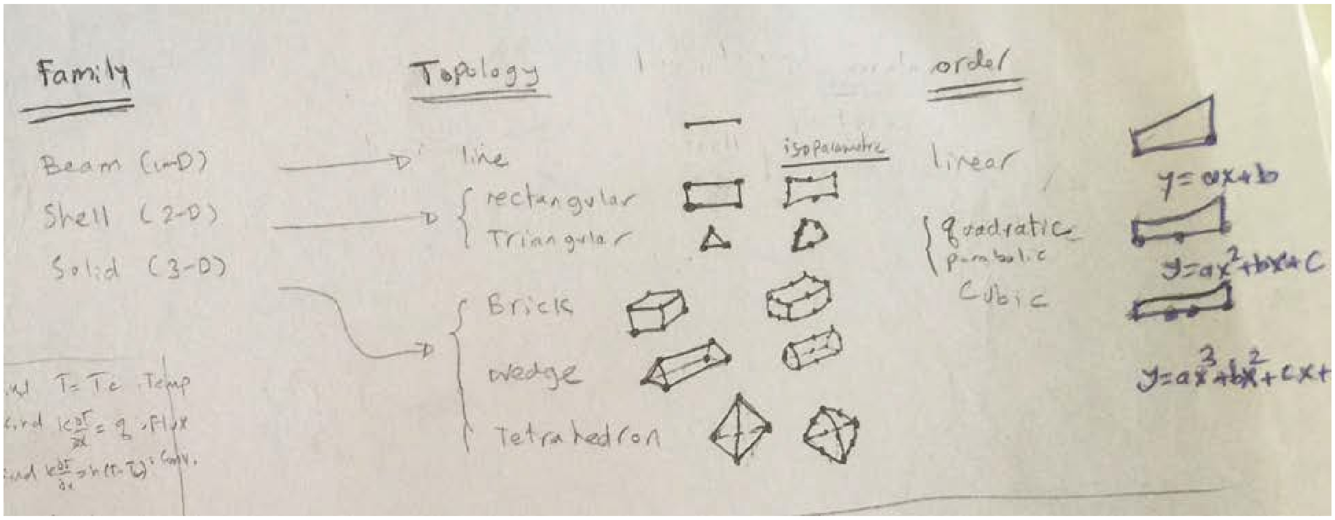
1. **Isotropic (Metals/Polymers/Ceramics)**
2. **Transversely Isotropic (Composite Plies)**
3. **Orthotropic (Smart Materials, Superconducting, Thin Films, etc)**
4. **Anisotropic (Often due to rotation of Trans. Iso. Or Orthotropic materials)**

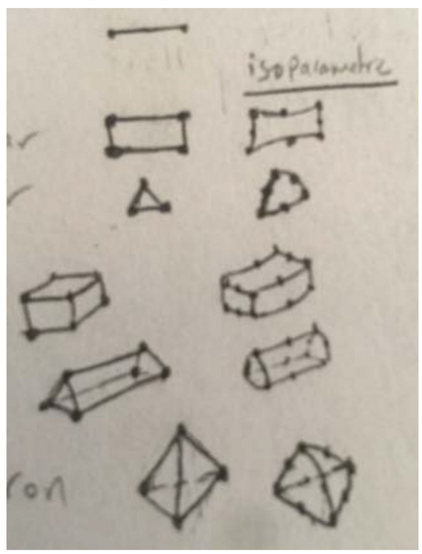
5. : V.1.5. Boundary Conditions (see BCs details):

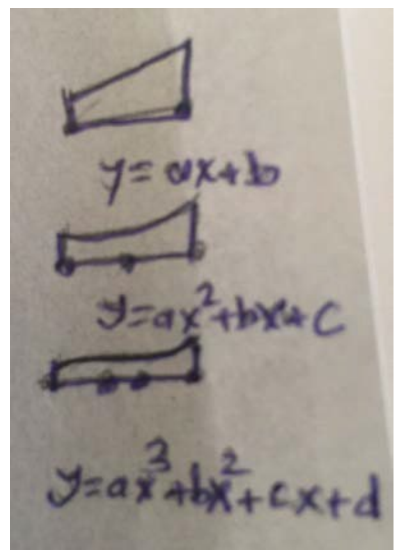
1. **Applied Loads**
2. **Restraints**

V.1.3. MESHING Details (Elements & Nodes)

SUMMARY OF MESHING Details (Elements & Nodes)







Family	**Topology**	**Order**
Beam (1-D)	Line	Linear
Shell (2-D)	Rectangular Triangular	Quadratic Parabolic
Solid (3-D)	Brick Wedge Tetrahedron	Cubic Cubic Cubic

V.1.5. Boundary Conditions Details

V.1.5.1. Boundary Conditions for Heat Transfer (HT) & Fluids (F)

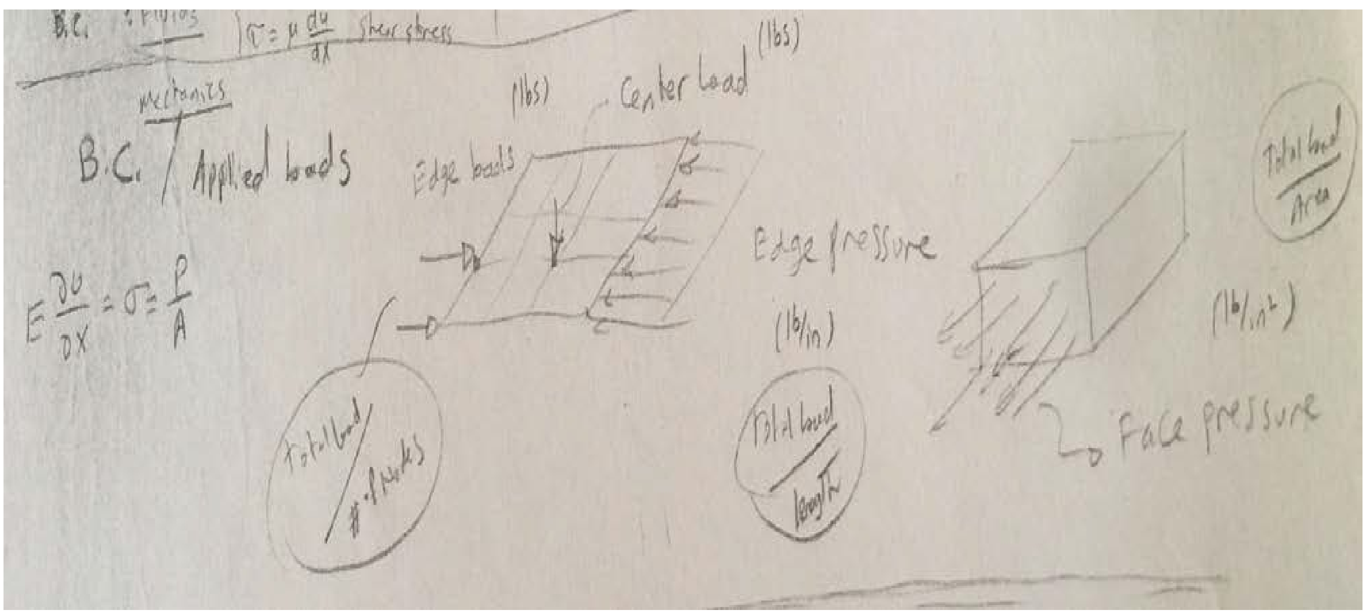
B.C. : Heat Transfer

$$\begin{cases} 1^{st} \text{ kind} & T = T_c \quad : \text{Temp} \\ 2^{nd} \text{ kind} & k\dfrac{\partial T}{\partial x} = q \quad : \text{Flux} \\ 3^{rd} \text{ kind} & k\dfrac{\partial T}{\partial x} = h(T - T_\infty) \quad : \text{Conv.} \end{cases}$$

B.C. : Fluids

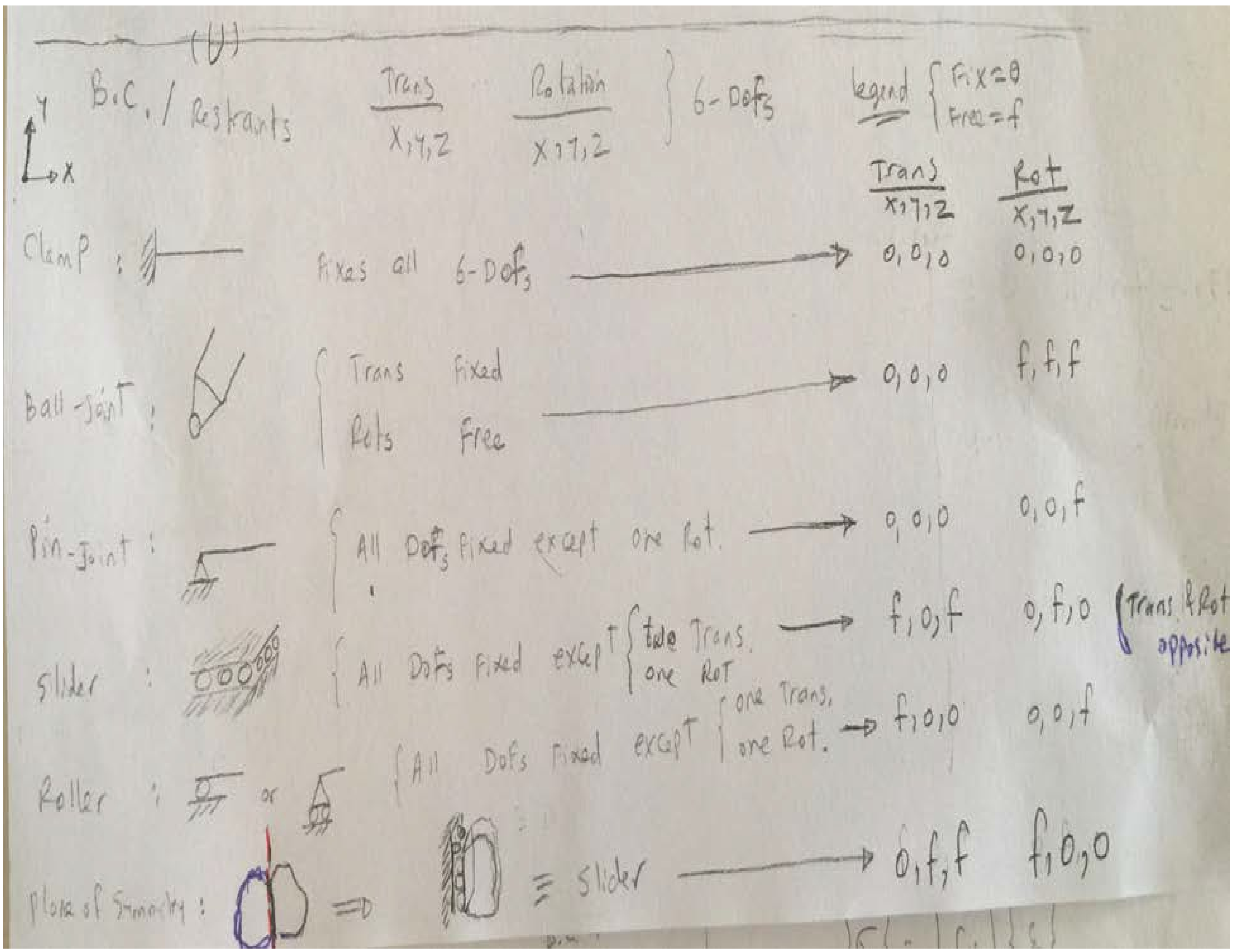
$$\begin{cases} U = U_c & \text{velocity} \\ \tau = \mu \dfrac{du}{dx} & \text{shear stress} \end{cases}$$

V.1.5.2. BCs: Applied Loads in Mechanics



V.1.5. Boundary Conditions Details

V.1.5.3. BCs: Restraints in Mechanics



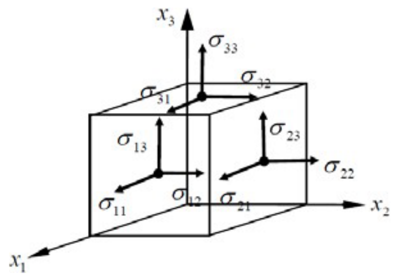
V. Finite Element Modeling & Analysis:

V.2. Step 2: Model Solution

V.2.1: Solve/Run

V.3. Step 3: Post-Processing

V.3: Output Selection (e.g., in Mechanics):



$$\left[\sigma_{ij}\right] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

$$\sigma ij = \sigma ji$$

$$\{\sigma_i\}_{6x1} = [C_{ij}]_{6x6}\, \{\varepsilon_i\}_{6x1}$$

V.3.1. Stresses (6 Independent Components, of a 2nd Order Tensor)

X-Normal (σ_{xx})	XY-Shear (σ_{xy})
Y-Normal (σ_{yy})	YZ-Shear (σ_{yz})
Z-Normal (σ_{zz})	ZX-Shear (σ_{zx})

V.3.2. Strain (6 Independent Components, of a 2nd Order Tensor)

X-Normal (ε_{xx})	XY-Shear (ε_{xy})
Y-Normal (ε_{yy})	YZ-Shear (ε_{yz})
Z-Normal (ε_{zz})	ZX-Shear (ε_{zx})

V.3.3. Displacements (3 Independent Components, of a Vector = 1st Order Tensor)

X-Direction (U_x)
Y-Direction (U_y)
Z-Direction (U_z)

V.3.4. Von Mises Stresses (Distorsion Energy Theory)

VM Stress in terms of **principle stresses**

$$\sigma_{VM} = \sqrt{\frac{\sigma_1 - \sigma_2{}^2 + \sigma_2 - \sigma_3{}^2 + \sigma_3 - \sigma_1{}^2}{2}}$$

VM Stress in terms of **6 components actual stresses**

$$\sigma_{VM} = \sqrt{\frac{(\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{yy} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{xx})^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)}{2}}$$

VI: Failure Analysis & Factor of Safety (FOS):

Two Considerations:

(I) Stress Critical Structures
(II) Stiffness (Deflection/Displacement) Critical Structures
(III) Materials Selection Issues Related to Failure Analysis

VI.I. Stress Critical Structures: FOS = $\sigma_{yield} / \sigma_{max} > 1$

VI.I.1. For Isotropic Materials (Metals, Polymers, Ceramics)

Use **Maximum Von Mises Stress:**

$$FOS = (\sigma_{yield} / \sigma_{max\ Von\ Mises}) > 1$$

VI.I.2. For Anisotropic Materials (Composite Materials)

Use **Maximum Stress Criterion**: i.e., find FOS for each stress-component (i.e., **FOS = direction-strength/maximum direction-stress**), and then the Min FOSs is the FOS.

For example:

(1) Find FOSs for ***Normal Stresses (3 componenets)***:

FOS for σ_{xx} : **FOS $|_{\sigma xx}$ = (Normal Strength in X-Direction / $\sigma_{xx\ max}$) > 1**

Likewise for σ_{yy} & σ_{zz} : FOS $|_{\sigma yy}$ & FOS $|_{\sigma zz} > 1$

(2) Find FOSs for ***Shear Stresses (3 independent componenets)***: FOS for σ_{xy} :

FOS $|_{\sigma xy}$ = (Shear Strength in XY-Plane / $\sigma_{xy\ max}$) > 1 Likewise for σ_{yz} & σ_{zx} : FOS $|_{\sigma yz}$ & FOS $|_{\sigma zx} > 1$

(3) Final FOS = Min (FOS $|_{\sigma xx}$, FOS $|_{\sigma yy}$, FOS $|_{\sigma zz}$, FOS $|_{\sigma xy}$, FOS $|_{\sigma yz}$, FOS $|_{\sigma zx}$) >1

Typical FOS in Design = 1.5, Human Life Involved= 2.5 (if confident about your assumption, modeling, and analysis); otherwise pick Larger (up to 4).

In **Space Applications** since weight is a critical factor, the analyses are conducted **accurately**, and hence it is afforded to use lower **FOSs = ~ 1.1 to 1.25**.

VI.I.3. Remedies if FOS = $\sigma_{yield} / \sigma_{max} < 1$:

We want to increase FOS to > 1: So,

1. **Either Increase the Numerator (i.e., $\sigma_{yield}\uparrow$)**
2. **Or Decrease the Denominator (i.e., $\sigma_{max}\downarrow$)**

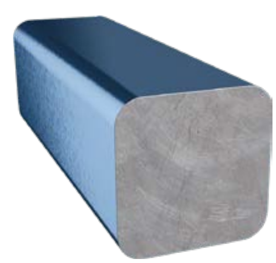
(1) Increase the Numerator (i.e., $\sigma_{yield}\uparrow$) :

By changing the material to a **Stronger Material** with higher σ_{yield} , e.g., change **Aluminum to Steel**.

(2) Decrease the Denominator (i.e., $\sigma_{max}\downarrow$):

σ can, in general, be modeled as $\sigma = F / A$

$$\sigma\downarrow = F\downarrow / A\uparrow$$



Since F is a design parameter and often cannot be changed, hence the remedy is to use a "beefier" components, i.e., with higher cross-sectional area, A, where stress is acting on.

For example: The beam with cross-section of **b=width**, **h=thickness**, and **l=length**

For **Axial Loading** (***Tension*** or ***Compression***): $\sigma\downarrow = F / A\uparrow$, Increase **A = b x h**

For **Lateral Loading** (***Bending***): $\sigma\downarrow = MC/I\uparrow$ & $I\uparrow = b\,h^3/12$,

Increase h since it has cubic effect in increasing I than b.

VI.II Stiffness (Deflection/Displacement) Critical Structures: Deflection/Displacement < δ_{limit}

VI.II.1. Limites of Deflections/Displacement in Designs:

For those structures that have **FOS >> 1**, then next consideration is if the material is stiff enough to give **Deflection/Displacement < δ_{limit} (say, not more than 10% of the thickness, to also stay within Linear Regime; e.g., assuming the nominal cross-section is 10mm x 10mm, then the δ_{limit} = 1 mm or less)**

Therefore, in many designs various components have their own ***Critical Deflection/Displacement Limits (δ_{limit})*** that designers should check with the sponsor/client; however, a **rule-of-thumb** is that:

Deflection/Displacement < 1 mm=0.040" (for a nominal cross-section: 10mm x 10mm to also stay within the Linear Regime)

VI.II.2. Remdeidies if Deflection/Displacement > δ_{limit}:

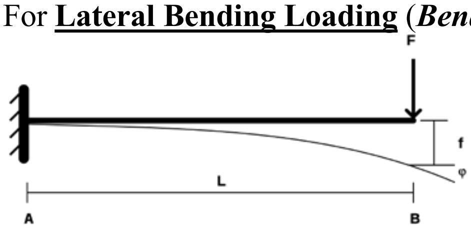
The amount of **Deflection/Displacement** in a structure is a function of the structure Stiffness (i.e., **Young's Modulus,** E: $\sigma = E\,\varepsilon$).

For **Axial Loading** (***Tension*** or ***Compression***): $\sigma = F/A = E\,\varepsilon = E\,\Delta L/L$:

$$F = EA\,\Delta L/L \Longrightarrow \quad \Delta L\downarrow = \delta\downarrow = F\downarrow L\downarrow / E\uparrow A\uparrow$$

Then, if F, L, and A are the design parameters that you cannot change, then the only remedy is to use a Stiffer Materials, e.g., change **Aluminum to Steel**

For **Lateral Bending Loading** (***Bending***):



$$f = \delta\downarrow = F\downarrow L^3\downarrow / 3\,E\uparrow I\uparrow$$, where $I = b\,h^3/12$

Understanding that L has a cubic effect on δ compared with F; however, they both may be the design parameters that you cannot change. In addition, EI is called "***Flexural Rigidity***" that should increase in this case. Again, note that to increase I, h has cubic effect in increasing I than b. Finally, to reduce δ, E (i.e., the materials stiffness) should increase, e.g., change **Aluminum to Steel**.

VI.III. Materials Selection Issues Related to Failure Analysis

If **Weight is a Major Factor in Design**, then:

VI.III.1. For Stress-Critical Structures:

For Stress-Critical Structures the parameter that should be maximized is not only σ_{yield} (i.e., ***Yield Strength***), but "σ_{yield} / ρ" (i.e., ***Specific Strength***) ***should be maximized,*** where ρ is the density of the material.

For example: (σ_{yield} / ρ) unit in the following is: "***KN*m/Kg***"

$(\sigma_{yield} / \rho)|_{\text{for C/E Composites}} = \sim \mathbf{2{,}000} >> (\sigma_{yield} / \rho)|_{\text{Steel}} = \sim \mathbf{200} > (\sigma_{yield} / \rho)|_{\text{Al}} = \sim\mathbf{100}$

VI.III.2. For Stiffness-Critical Structures:

For Stiffness-Critical Structures the parameter that should be maximized is not only E (i.e., ***Young's Modulus or Stiffness***), but "E / ρ" (i.e., ***Specific Stiffness***), ***should be maximized,*** where ρ is the density of the material.

For example: (E / ρ) unit is in "$\boldsymbol{m^2\, s^{-2} * 10^6}$".-

$(E / \rho)|_{\text{for C/E Composites}} = \sim \mathbf{100} >> (E / \rho)|_{\text{Steel}} = \sim \mathbf{25} = (E / \rho)|_{\text{Al}} = \sim\mathbf{25}$