FEM/FEA Boundary Conditions, and Failure Analysis

Three Cases must use FEM/FEA:

(1) Irregular Geometry
(2) Complex Material Properties
(3) Nonlinearity

NOTE: Three general cases you would need to use FEA for your Analysis and Analytical Solutions do not apply:

(1) Irregular Geometry,
(2) Complex and Non-homogenous Materials Properties, and
(3) Nonlinearity of the Problems

I: Classical 1-D FEA Problem
(Axial Loading of a Bar)

https://enterfea.com/finite-element-analysis-by-hand/
\[ F = K \Delta X \], where \( K = \frac{AE}{L} \) & \( \Delta X = \Delta U \)

\[ \frac{F}{A} = G = E \cdot \varepsilon = E \cdot \frac{\Delta L}{L} = E \cdot \frac{u_2 - u_1}{L} \]

\[ F = \frac{AE}{L} (u_2 - u_1) \]
AE/L = K

\[
\begin{bmatrix}
  k & -k \\
  -k & k
\end{bmatrix}
\begin{bmatrix}
  u_1 \\
  u_2
\end{bmatrix} =
\begin{bmatrix}
  f_1 \\
  f_2
\end{bmatrix}
\]

\[
\{f_1, f_2\} = [k_e] \{u_1, u_2\}
\]

\[
[k_e] = \begin{bmatrix}
  k & -k \\
  -k & k
\end{bmatrix}
\]

Pre-Dot the above Eq by \([K]^{-1}\)

\[
[K]^{-1} \{F\} = [K]^{-1} [K] \{U\}, \text{ where } [K]^{-1} [K] = [I] \implies [K]^{-1} \{F\} = \{U\}
\]

\[
\{U_i\} = [K_{ij}]^{-1} \{F_j\}
\]

\[
\{\varepsilon_i\} = \left[\frac{d}{dx_{ij}}\right] \{u_j\}, \quad \{\sigma_i\}_{6x1} = [C_{ij}]_{6x6} \{\varepsilon_i\}_{6x1}
\]

**General 3D Case:**

\[
\sigma_i = \sigma_x, \sigma_y, \sigma_z, \sigma_{xy}, \sigma_{yz}, \sigma_{zx} \quad \& \quad \sigma_{ij} = \sigma_{ji}
\]

\[
\varepsilon_i = \varepsilon_x, \varepsilon_y, \varepsilon_z, \varepsilon_{xy}, \varepsilon_{yz}, \varepsilon_{zx} \quad \& \quad \varepsilon_{ij} = \varepsilon_{ji}
\]
II: 3D General Materials Properties in Mechanics

\[ C_{ij} = C_{ji} = 21 \] independent materials Anisotropic properties \([C_{ij}]\)

= 9 independent materials Orthotropic properties (3E’s, 3Gs, 3\(\nu\)s)

= 5 independent materials Transversly-Isotropic properties (2Es, 1G, 2\(\nu\)s)

= 2 independent materials Isotropic properties (E & \(\nu\), since \(G = E/(2(1+\nu))\))

III. Geometries

For the following Regular Geometries Analytical Solutions and associated Governing Equations (written for the given coordinate system) plus the boundary and initial conditions can be used. Otherwise (i.e., Irregular Geometries) FEA should be used.

(III.1): Cartesian/Orthogonal Coordinate System: \(x, y, z\)

\[ dV = dx \, dy \, dz \]

Volume element in Cartesian coordinates

Heat Conduction Eq in Cartesian Coordinate System:

\[ \frac{\partial^2 T}{\partial x^2} + \frac{\dot{e}_{\text{gen}}}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \]
(III.2): Cylindrical/Orthogonal Coordinate System: \( \rho, \phi, z \)

\[ dV = \rho d\phi d\rho dz \]

Volume element for a cylinder

Unit Vectors

Heat Conduction Eq in Cylindrical Coordinate System:

\[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \frac{\dot{e}_{\text{gen}}}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \]

(III.3): Spherical/Orthogonal Coordinate System: \( r, \phi, \theta \)

\[ dV = r^2 \sin \theta d\theta d\phi dr \]

Spherical coordinates

Unit Vectors
Heat Conduction Eq in Spherical Coordinate System:

\[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial T}{\partial r} \right) + \frac{\dot{\varepsilon}_{\text{gen}}}{\dot{\varepsilon}} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \]

IV. Nonlinearity:

IV.1. Nonlinearity due to Small Deformation versus Large Deformation

If \( f \) is the tip \textbf{Deflection} and \( t \) is the \textbf{thickness} of a beam, then:

If \( f \ll t \), then \( f \) is considered “\textit{small deformation}” and the problem is \textbf{Linear}.

If \( f \gg t \), then \( f \) is considered “\textit{large deformation}” and the problem is \textbf{Nonlinear}.

IV.2. Nonlinearity due to the Governing Equations

If the Field/Dependent Variable within the Individual Terms of a Governing Equation is to the \textbf{first power} only, then the problem is \textbf{Linear}.

If the Field/Dependent Variable within the Individual Terms of a Governing Equation is to the \textbf{second or higher power}, then the problem is \textbf{Nonlinear}.

If \( K \) is variable in space, the 3D Heat Conduction Eq is:

\[ \frac{\partial}{\partial x} \left( k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left( k \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left( k \frac{\partial T}{\partial z} \right) + \dot{\varepsilon}_{\text{gen}} = \rho c \frac{\partial T}{\partial t} \]
If $K$ is constant in space and the material is **Isotropic**, the 3D Heat Conduction Eq is (where, $\alpha = k/\rho c$: Thermal Diffusivity):

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{e}_{\text{gen}}}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

If $K$ is constant in space and the material is **Orthotropic**, the 3D Heat Conduction Eq is:

$$K_x \left( \frac{\partial^2 T}{\partial x^2} \right) + K_y \left( \frac{\partial^2 T}{\partial y^2} \right) + K_z \left( \frac{\partial^2 T}{\partial z^2} \right) + (e_{\text{gen}}) = \rho c \left( \frac{\partial T}{\partial t} \right)$$

The above Eqs are **Linear** since for each term of the Eq, the Field/Dependent Variable (i.e., $T$) is to the **first power**.

If the Thermal Conductivity components are functions of Temperature, then the above equation becomes **Nonlinear** since it will have the Field/Dependent Variable (i.e., $T$) to the **second power** for each term of the Eq. as:

$$K_x(t) \left( \frac{\partial^2 T}{\partial x^2} \right) + K_y(t) \left( \frac{\partial^2 T}{\partial y^2} \right) + K_z(t) \left( \frac{\partial^2 T}{\partial z^2} \right) + (e_{\text{gen}}) = \rho c \left( \frac{\partial T}{\partial t} \right)$$

**Governing Eq for Bending of a Beam:**

$$EI \frac{d^4 w}{dx^4} = q(x)$$
\[ \frac{d^3w}{dx^3} = \frac{q.x}{EA} + C_1 \]
\[ \frac{d^2w}{dx^2} = \frac{q.x^2}{EA2} + C_1x + C_2 \]
\[ \frac{dw}{dx} = \frac{q.x^3}{EA6} + \frac{C_1x^2}{2} + C_2x + C_3 \]
\[ \frac{dw}{dx} = \frac{q.x^4}{EA24} + \frac{C_1x^3}{6} + \frac{C_2x^2}{2} + C_3x + C_4 \]

\[ w(x) = \frac{q.x}{24EA}(l^3 - 2lx^2 + x^3) \]

**Governing Eq for Bending of a Plate:**

\[ \frac{\partial^4w}{\partial x^4} + 2\frac{\partial^4w}{\partial x^2 \partial y^2} + \frac{\partial^4w}{\partial y^4} = -\frac{q}{D} \]

Where:

\[ D = \frac{Eh^3}{12(1-\nu^2)} \]
V. Finite Element Modeling & Analysis:

Any Finite Element Modeling (FEM) and Finite Element Analysis (FEA) includes the following three main steps: (1) Pre-Processing, (2) Model Solution, (3) Post-Processing.

V.1. Step 1: Pre-Processing

1. **V.1.1. Object Generation:**
   1. 3D: **3D** Solid Modeling with **Solid Elements**
   2. 1D & 2D: **1D Beam Element** & **2D Shell Elements**

2. **V.1.2. Pick Analysis Type (for Step 2: Model Solution):**
   1. Structural Analysis (**Static** or **Dynamic; Linear** or **Non-Linear**)
   2. Heat Transfer Analysis (**Steady State** or **Transient; Linear** or **Non-Linear**)
   3. Fluids Analysis (**Steady State** or **Transient; Linear** or **Non-Linear**)

3. **V.1.3. Meshing (see Meshing details):**
   1. Select Elements
   2. Select Nodes

4. **V.1.4. Material Properties (see [Cij] details):**
   1. Isotropic (Metals/Polymer/Ceramics)
   2. Transversely Isotropic (Composite Plies)
   3. Orthotropic (Smart Materials, Superconducting, Thin Films, etc)
   4. Anisotropic (Often due to rotation of Trans. Iso. Or Orthotropic materials)

5. **V.1.5. Boundary Conditions (see BCs details):**
   1. Applied Loads
   2. Restraints
### V.1.3. MESHING Details (Elements & Nodes)

**SUMMARY OF MESHING Details (Elements & Nodes)**

<table>
<thead>
<tr>
<th>Family</th>
<th>Topology</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>Beam (1-D)</td>
<td>Line</td>
<td>Linear</td>
</tr>
<tr>
<td>Shell (2-D)</td>
<td>Rectangular</td>
<td>Quadratic</td>
</tr>
<tr>
<td></td>
<td>Triangular</td>
<td>Parabolic</td>
</tr>
<tr>
<td>Solid (3-D)</td>
<td>Brick</td>
<td>Cubic</td>
</tr>
<tr>
<td></td>
<td>Wedge</td>
<td>Cubic</td>
</tr>
<tr>
<td></td>
<td>Tetrahedron</td>
<td>Cubic</td>
</tr>
</tbody>
</table>
V.1.5. Boundary Conditions Details

V.1.5.1. Boundary Conditions for Heat Transfer (HT) & Fluids (F)

V.1.5.2. BCs: Applied Loads in Mechanics
V.1.5. Boundary Conditions Details

V.1.5.3. BCs: Restraints in Mechanics

[Handwritten notes and diagrams explaining boundary conditions, including types of restraints such as clamp, ball joint, pin joint, slider, and roller, with corresponding degrees of freedom (DOFs) and constraints.]
V. Finite Element Modeling & Analysis:

V.2. Step 2: Model Solution

V.2.1: Solve/Run
V.3. Step 3: Post-Processing

V.3: Output Selection (e.g., in Mechanics):

V.3.1. Stresses (6 Independent Components, of a 2nd Order Tensor)

- X-Normal ($\sigma_{xx}$)
- Y-Normal ($\sigma_{yy}$)
- Z-Normal ($\sigma_{zz}$)
- XY-Shear ($\sigma_{xy}$)
- YZ-Shear ($\sigma_{yz}$)
- ZX-Shear ($\sigma_{zx}$)

\[
\begin{bmatrix}
\sigma_{ij}
\end{bmatrix} = \begin{bmatrix}
\sigma_{11} & \sigma_{12} & \sigma_{13} \\
\sigma_{21} & \sigma_{22} & \sigma_{23} \\
\sigma_{31} & \sigma_{32} & \sigma_{33}
\end{bmatrix}
\]

\[\sigma_{ij} = \sigma_{ji}\]

\[
\{\sigma_i\}_{6x1} = [C_{ij}]_{6x6} \{\varepsilon_i\}_{6x1}
\]

V.3.2. Strain (6 Independent Components, of a 2nd Order Tensor)

- X-Normal ($\varepsilon_{xx}$)
- Y-Normal ($\varepsilon_{yy}$)
- Z-Normal ($\varepsilon_{zz}$)
- XY-Shear ($\varepsilon_{xy}$)
- YZ-Shear ($\varepsilon_{yz}$)
- ZX-Shear ($\varepsilon_{zx}$)

V.3.3. Displacements (3 Independent Components, of a Vector = 1st Order Tensor)

- X-Direction ($U_x$)
- Y-Direction ($U_y$)
- Z-Direction ($U_z$)

V.3.4. Von Mises Stresses (Distorsion Energy Theory)

**VM Stress** in terms of principle stresses

\[
\sigma_{VM} = \sqrt{\frac{\sigma_1 - \sigma_2}{2}^2 + \frac{\sigma_2 - \sigma_3}{2}^2 + \frac{\sigma_3 - \sigma_1}{2}^2}
\]

**VM Stress** in terms of 6 components actual stresses

\[
\sigma_{VM} = \sqrt{\frac{(\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{yy} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{xx})^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)}{2}}
\]
VI: Failure Analysis & Factor of Safety (FOS):

Two Considerations:

(I) Stress Critical Structures
(II) Stiffness (Deflection/Displacement) Critical Structures
(III) Materials Selection Issues Related to Failure Analysis

VI.I. Stress Critical Structures: FOS = $\frac{\sigma_{\text{yield}}}{\sigma_{\text{max}}} > 1$

VI.I.1. For Isotropic Materials (Metals, Polymers, Ceramics)

Use Maximum Von Mises Stress:

$$\text{FOS} = \left(\frac{\sigma_{\text{yield}}}{\sigma_{\text{max Von Mises}}}\right) > 1$$

VI.I.2. For Anisotropic Materials (Composite Materials)

Use Maximum Stress Criterion: i.e., find FOS for each stress-component (i.e., FOS = direction-strength/maximum direction-stress), and then the Min FOSs is the FOS.

For example:

1. Find FOSs for Normal Stresses (3 components):
   FOS for $\sigma_{xx}$: $\text{FOS} \mid \sigma_{xx} = \frac{\text{Normal Strength in X-Direction}}{\sigma_{xx \text{ max}}} > 1$
   Likewise for $\sigma_{yy}$ & $\sigma_{zz}$: $\text{FOS} \mid \sigma_{yy} \& \text{FOS} \mid \sigma_{zz} > 1$

2. Find FOSs for Shear Stresses (3 independent components): FOS for $\sigma_{xy}$:
   $\text{FOS} \mid \sigma_{xy} = \frac{\text{Shear Strength in XY-Plane}}{\sigma_{xy \text{ max}}} > 1$
   Likewise for $\sigma_{yz} \& \sigma_{zx}$: $\text{FOS} \mid \sigma_{yz} \& \text{FOS} \mid \sigma_{zx} > 1$

3. Final FOS = Min (FOS $\mid \sigma_{xx}$, FOS $\mid \sigma_{yy}$, FOS $\mid \sigma_{zz}$, FOS $\mid \sigma_{xy}$, FOS $\mid \sigma_{yz}$, FOS $\mid \sigma_{zx}$) $> 1$

Typical FOS in Design = 1.5, Human Life Involved = 2.5 (if confident about your assumption, modeling, and analysis); otherwise pick Larger (up to 4).

In Space Applications since weight is a critical factor, the analyses are conducted accurately, and hence it is afforded to use lower FOSs $\approx 1.1$ to $1.25$. 


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VI.I.3. Remedies if FOS = $\sigma_{\text{yield}} / \sigma_{\text{max}} < 1$:

We want to increase FOS to $> 1$: So,

(1) Either Increase the Numerator (i.e., $\sigma_{\text{yield}}$)

(2) Or Decrease the Denominator (i.e., $\sigma_{\text{max}}$)

(1) **Increase the Numerator (i.e., $\sigma_{\text{yield}}$):**

By changing the material to a **Stronger Material** with higher $\sigma_{\text{yield}}$, e.g., change **Aluminum to Steel**.

(2) **Decrease the Denominator (i.e., $\sigma_{\text{max}}$:**

$\sigma$ can, in general, be modeled as $\sigma = F / A$

\[ \sigma = F / A \]

Since $F$ is a design parameter and often cannot be changed, hence the remedy is to use a “beefier” components, i.e., with higher cross-sectional area, $A$, where stress is acting on.

**For example**: The beam with cross-section of $b=$width, $h=$thickness, and $l=$length

For **Axial Loading** (Tension or Compression): $\sigma = F / A$, Increase $A = b \times h$

For **Lateral Loading** (Bending): $\sigma = MC/I$, $I = bh^3/12$, Increase $h$ since it has cubic effect in increasing $I$ than $b$. 

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VI. II  Stiffness (Deflection/Displacement) Critical Structures: 
Deflection/Displacement < $\delta_{\text{limit}}$

VI. II.1. Limits of Deflections/Displacement in Designs:

For those structures that have $\text{FOS} >> 1$, then next consideration is if the material is stiff enough to give Deflection/Displacement $< \delta_{\text{limit}}$ (say, not more than 10% of the thickness, to also stay within Linear Regime; e.g., assuming the nominal cross-section is 10mm x 10mm, then the $\delta_{\text{limit}} = 1 \text{ mm or less}$)

Therefore, in many designs various components have their own Critical Deflection/Displacement Limits ($\delta_{\text{limit}}$) that designers should check with the sponsor/client; however, a rule-of-thumb is that:

Deflection/Displacement $< 1 \text{ mm}=0.040”$ (for a nominal cross-section: 10mm x 10mm to also stay within the Linear Regime)

VI. II.2. Remedies if Deflection/Displacement $> \delta_{\text{limit}}$:

The amount of Deflection/Displacement in a structure is a function of the structure Stiffness (i.e., Young’s Modulus, $E$: $\sigma = E \varepsilon$).

For Axial Loading (Tension or Compression): $\sigma = F / A = E \varepsilon = E \Delta L / L$:

$F = EA \Delta L / L \implies \Delta L = \delta = F L / E A$

Then, if $F$, $L$, and $A$ are the design parameters that you cannot change, then the only remedy is to use a Stiffer Materials, e.g., change Aluminum to Steel

For Lateral Bending Loading (Bending):

$\delta = \frac{F L^3}{3EI}$, where $I = bh^3/12$

Understanding that $L$ has a cubic effect on $\delta$ compared with $F$; however, they both may be the design parameters that you cannot change. In addition, $EI$ is called “Flexural Rigidity” that should increase in this case. Again, note that to increase $I$, $h$ has cubic effect in increasing $I$ than $b$. Finally, to reduce $\delta$, $E$ (i.e., the materials stiffness) should increase, e.g., change Aluminum to Steel.
VI.III. Materials Selection Issues Related to Failure Analysis

If **Weight is a Major Factor in Design**, then:

**VI.III.1. For Stress-Critical Structures:**

For Stress-Critical Structures the parameter that should be maximized is not only $\sigma_{\text{yield}}$ (i.e., Yield Strength), but **$\sigma_{\text{yield}} / \rho$** (i.e., Specific Strength) **should be maximized**, where $\rho$ is the density of the material.

**For example:** $(\sigma_{\text{yield}} / \rho)$ unit in the following is: “KN*m/Kg”

$(\sigma_{\text{yield}} / \rho)|_{\text{C/E Composites}} = 2,000 > (\sigma_{\text{yield}} / \rho)|_{\text{Steel}} = 200 > (\sigma_{\text{yield}} / \rho)|_{\text{Al}} = 100$

**VI.III.2. For Stiffness-Critical Structures:**

For Stiffness-Critical Structures the parameter that should be maximized is not only $E$ (i.e., Young’s Modulus or Stiffness), but **$E / \rho$** (i.e., Specific Stiffness), **should be maximized**, where $\rho$ is the density of the material.

**For example:** $(E / \rho)$ unit is “m²s⁻² * 10⁶”.

$(E / \rho)|_{\text{C/E Composites}} = 100 > (E / \rho)|_{\text{Steel}} = 25 = (E / \rho)|_{\text{Al}} = 25$